M303 Further Pure Mathematics

Exam 8 June 2018

# Unofficial Solutions - these have NOT been checked by the module team. They will likely contain mistakes and omissions but may be helpful for revision purposes.

(Student version)


Collaboration by Contributors to M303 Forum discussions

2018

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$5^{\text{th}}$ May 2021

Q 1.

(a)

(i) Applying the Euclidean Algorithm we have:

$$\begin{aligned} 336 &= 2 \times 147 + 42 \\ 147 &= 3 \times 42 + 21 \\ 42 &= 2 \times 21 + 0 \end{aligned}$$

So $\text{hcf}(336, 147) = 21$ (since this is the last non-zero remainder).

(ii) We seek a solution to the equation: $147s + 336t = 21$, so we use the equations above to express 21 as a product of 147 and 336 as follows:

$$\begin{aligned} 21 &= 147 - 3 \times 42 \\ &= 147 - 3 \times (336 - 2 \times 147) \\ &= 7 \times 147 - 3 \times 336 \end{aligned}$$

Therefore $s = 7$ and $t = -3$

(b) We use the Principle of Mathematical Induction (Handbook, p14). Let $P(n)$ be the statement:

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

We check P(1) is true: For $n = 1$, LHS of $P(1) = 1 \times 2 = 2$,
RHS of $P(1) = \dfrac{1 \times 2 \times 3}{3} = 2$.
So $P(1)$ is true, and forms the basis for induction.
Now, assume that $P(k)$ is true for some $k \in \mathbb{N}$, that is:
$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + k(k+1) = \dfrac{k(k+1)(k+2)}{3}$.
Then we want to show that $P(k+1)$ is true, that is:

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}.$$

$$\begin{aligned} \text{LHS} = 1 \times 2 + 2 \times 3 + 3 \times 4 + \ldots + (k+1)(k+2) &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \text{ by assumption} \\ &= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \\ &= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\ &= \frac{(k+1)(k+2)(k+3)}{3} = \text{RHS} \end{aligned}$$

So $P(k+1)$ is true.
Thus $P(n)$ is true for all $n$, by the Principle of Mathematical Induction.

Q 2.

(a)

(i) $|G| = 12$.

" Write down" so no reason needed, but Prop 2.10 HB p 32 applies.

(ii) 4

(iii) Either (1, 2) or (2,2) is correct.

(iv) Yes. By Thm. 2.11 (HB p32) because 3 and 4 are coprime.

(b)

(i) $G \cong \mathbb{Z}_3 \times \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$,
$H \cong \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.
Since $G$ and $H$ are not isomorphic to the same direct product of cyclic groups of prime-power order, they are not isomorphic.

(ii) No. Neither $G$ nor $H$ is cyclic, but $\mathbb{Z}_{1200}$ is cyclic.

(iii) No, because $D_{600}$ is not Abelian, but $G$ and $H$ are.

Q 3.

(a) The divisors of 60 are $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$.

(i) There are 12 divisors, so $\tau(60) = 12$.

(ii) Using Appendix 2, HB p57, we have:

$$\begin{aligned} \sigma(60) &= \sigma(3) \times \sigma(4) \times \sigma(5) \\ &= 4 \times 7 \times 6 \\ &= 168 \end{aligned}$$

(iii) By Alternative Formula below Proposition 4.2 HB Page 44,

$$\phi(60) = \phi(3) \times \phi(4) \times \phi(5) = 2 \times 2 \times 4 = 16.$$

(b)

(i) We have $223 \equiv 7 \pmod 8$, so $(2/223) = 1$ by Proposition 3.2, (HB p46).

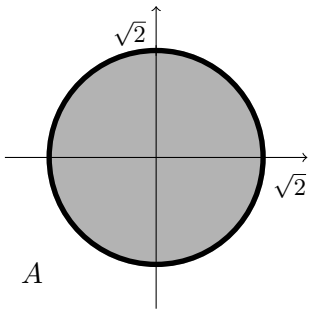
(ii)

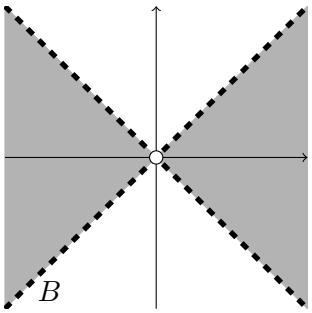
| $(179/223)$ | Rule used, HB p46 |
|---|---|
| $= -(223/179)$ | Thm 4.2 |
| $= -(44/179)$ | 2.2(a) |
| $= -(4/179)(11/179)$ | 2.2(c) |
| $= -(11/179)$ | 2.2(b) |
| $= (179/11)$ | 4.2 |
| $= (3/11)$ | 2.2(a) |
| $= 1$ | 4.7 |

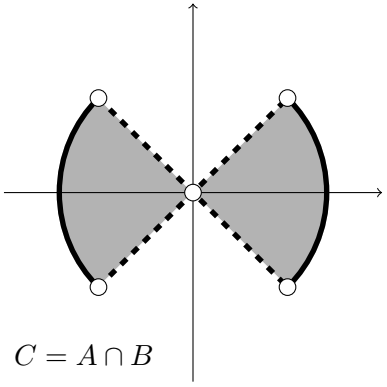
(c) By the Rational Root Test, any rational root of $f(x)$ will have the form $\frac{p}{q} \in \mathbb{Q}$ where $p \mid -2$ and $q \mid 1$.
Thus possible roots are $1, -1, 2, -2$.
Checking these in turn:
$f(1) = 1 - 1 + 3 - 2 = 1 \neq 0$,
$f(-1) = -1 - 1 - 3 - 2 = -7 \neq 0$,
$f(2) = 8 - 4 + 6 - 2 = 8 \neq 0$,
$f(-2) = -8 - 4 - 6 - 2 = -20 \neq 0$,
We conclude that $f(x)$ has no rational root.
But since $f(x)$ has degree 3, if it reducible it will have a linear factor.
We therefore conclude that $f(x)$ is irreducible over $\mathbb{Q}$.

Q 4.

(a)




(b) Any open ball with centre $(1,\, 0)$ and radius less than $\sqrt{2}-1$ will lie inside $C$. Thus $(1, 0)$ is an interior point of $C$ for the Euclidean metric.

(c)

$$\begin{aligned}
\mathrm{Cl}_{(\mathbb{R}^2,d^{(2)})}(C) =& \{(a,b)\in\mathbb{R}^2 : a^2+b^2 \leq 2\} \cap \{(a,b)\in\mathbb{R}^2 : |b| \leq |a|\} \\
\mathrm{Int}_{(\mathbb{R}^2,d^{(2)})}(C) =& \{(a,b)\in\mathbb{R}^2 : a^2+b^2 < 2\} \cap \{(a,b)\in\mathbb{R}^2 : |b| < |a|\} \\
\mathrm{Bd}_{(\mathbb{R}^2,d^{(2)})}(C) =& \left\{(a,b)\in\mathbb{R}^2 : a^2+b^2 = 2, |a| \geq 1\right\} \cup \{(a,b)\in\mathbb{R}^2 : |a| = |b|\,, |a| < 1\}
\end{aligned}$$

(d) Since the closure of $C$ is $d^{(2)}$- closed and bounded, by the Heine-Borel Theorem, it is $d^{(2)}$-compact.

Q 5.

We are given $\left[\mathbb{Q}\left(\sqrt{6}\right):\mathbb{Q}\right]=\left[\mathbb{Q}\left(\sqrt{10}\right):\mathbb{Q}\right]=2$ and $\sqrt{10}\notin\mathbb{Q}\left(\sqrt{6}\right)$, $\sqrt{6}\notin\mathbb{Q}\left(\sqrt{10}\right)$.

(a) $\alpha=\sqrt{6}-\sqrt{10}$. Therefore:

$$\begin{aligned}\alpha^3&=\left(\sqrt{6}-\sqrt{10}\right)^3\\&=(\sqrt{6})^3+3(\sqrt{6})^2\left(-\sqrt{10}\right)+3\sqrt{6}\left(-\sqrt{10}\right)^2+\left(-\sqrt{10}\right)^3\\&=6\sqrt{6}-18\sqrt{10}+30\sqrt{6}-10\sqrt{10}\\&=36\sqrt{6}-28\sqrt{10}.\end{aligned}$$

Therefore $a=36$ and $b=-28$ (NOTE: $a,b\in\mathbb{Q}$).

(b) Consider $\alpha^3-36\alpha$:

$$\begin{aligned}\alpha^3-36\alpha&=36\sqrt{6}-28\sqrt{10}-36\sqrt{6}+36\sqrt{10})\\&=8\sqrt{10}\in\mathbb{Q}(\alpha).\end{aligned}$$

And since $1/8\in\mathbb{Q}(\alpha)$, we have $\sqrt{10}\in\mathbb{Q}(\alpha)$.
Further,
$\alpha+\sqrt{10}=\sqrt{6}-\sqrt{10}+\sqrt{10}=\sqrt{6}\Rightarrow\sqrt{6}\in\mathbb{Q}(\alpha)$.

(c) Since $\sqrt{6},\sqrt{10}\in\mathbb{Q}(\alpha)$ ( by part (b)) it follows that
$\mathbb{Q}(\sqrt{6},\sqrt{10})\subseteq\mathbb{Q}(\alpha)$.
Also since, $\alpha=\sqrt{6}-\sqrt{10}$ is a linear combination of elements in $\mathbb{Q}(\sqrt{6},\sqrt{10})$ it follows that $\mathbb{Q}(\alpha)\subseteq\mathbb{Q}(\sqrt{6},\sqrt{10})$.
Thus $\mathbb{Q}(\sqrt{6},\sqrt{10})=\mathbb{Q}(\alpha)$.

(d) Since $\sqrt{6}\notin\mathbb{Q}(\sqrt{10})$ the minimal polynomial for $\sqrt{6}$ over $\mathbb{Q}(\sqrt{10})$ is $x^2-6$. Hence $[\mathbb{Q}(\sqrt{6},\sqrt{10}):\mathbb{Q}(\sqrt{10})]=2$.
By part (c) and the KLM Theorem

$$\begin{aligned}[\mathbb{Q}(\alpha):\mathbb{Q}]&=[\mathbb{Q}(\sqrt{6},\sqrt{10}):\mathbb{Q}]\\&=[\mathbb{Q}(\sqrt{6},\sqrt{10}):\mathbb{Q}(\sqrt{10})]\times[\mathbb{Q}(\sqrt{10}):\mathbb{Q}]\\&=2\times2=4\end{aligned}$$

The rôles of $\sqrt{6}$ and $\sqrt{10}$ may be interchanged here.

(e) $\alpha^2=\left(\sqrt{6}-\sqrt{10}\right)^2=6-2\sqrt{60}+10=16-2\sqrt{60}$
and so $\alpha^2-16=-2\sqrt{60}$ giving $(\alpha^2-16)^2=\left(-2\sqrt{60}\right)^2=240$.
Hence $\alpha^4-32\alpha^2+256=240$, that is $\alpha^4-32\alpha^2+16=0$.
Since, by part(d), the minimal polynomial for $\alpha$ over $\mathbb{Q}$ must have degree 4, it follows that $x^4-32x^2+16$ is the required minimal polynomial. (It is clearly monic.)

Q 6.

(a)

"Write down", so simply writing "2" gets the mark.

$$\begin{aligned} d\left((0,1),(1,0)\right) &= \max\left\{2\left|0-1\right|,\left|1-0\right|\right\} \\ &= \max\left\{2,1\right\} \\ &= 2 \end{aligned}$$

(b) A $d$-disconnection of $X$ is $\{U,V\}$ where
$U=\left\{(a,b)\in\mathbb{R}^2 : a^2+b^2>2\right\}$; $\quad V=\left\{(a,b)\in\mathbb{R}^2 : a^2+b^2<2\right\}$.

(c) Let $(\mathbf{a}_n)=\left(\left(\frac{1}{n},\frac{\sin n}{n}\right)\right)_{n=1}^{\infty}$; we guess that $(\mathbf{a}_n)$ is $d$-convergent to $(0,0)$. Now

$$\begin{aligned} d\left(\left(\frac{1}{n},\frac{\sin n}{n}\right),(0,0)\right) &= \max\left\{2\left|\frac{1}{n}-0\right|,\left|\frac{\sin n}{n}-0\right|\right\} \\ &= \max\left\{\frac{2}{n},\left|\frac{\sin n}{n}\right|\right\} \\ &\leq \frac{2}{n} \end{aligned}$$

Also, $\left(\frac{2}{n}\right)$ is a multiple of a Basic null sequence so, as $0\leq d\left(\mathbf{a}_n,(0,0)\right)\leq\frac{2}{n}$, it follows, by the Squeeze Rule, that $\left(d\left(\mathbf{a}_n,(0,0)\right)\right)$ is null.

Hence $(\mathbf{a}_n)=\left(\left(\frac{1}{n},\frac{\sin n}{n}\right)\right)_{n=1}^{\infty}$ is $d$-convergent.

(d) Let $(\mathbf{b}_n)=\left(\left(\sqrt{2}+\frac{1}{n},0\right)\right)_{n=1}^{\infty}$; we will show that is $d$-Cauchy but not $d$-convergent.
*$(\mathbf{b}_n)$ is $d$-Cauchy*
Take any $\varepsilon>0$ and choose $N\in\mathbb{N}$ with $N>4/\varepsilon$. Then

$$\begin{aligned} d(\mathbf{b}_m,\mathbf{b}_n) &= \max\left\{2\left|\sqrt{2}+\frac{1}{m}-\sqrt{2}-\frac{1}{n}\right|,0\right\} \\ &= \max\left\{2\left|\sqrt{2}+\frac{1}{m}-\sqrt{2}-\frac{1}{n}\right|,0\right\} \\ &= \max\left\{2\left|\frac{1}{m}-\frac{1}{n}\right|,0\right\} \\ &\leq 2\left(\frac{1}{m}+\frac{1}{n}\right) \end{aligned}$$

Now if $m,n>N$ the latter is $<4/N<\varepsilon$, that is $(\mathbf{b}_n)$ is $d$-Cauchy.
*$(\mathbf{b}_n)$ is not $d$-convergent*
We give a proof by contradiction.
Suppose then that $(\mathbf{b}_n)$ $d$-converges to some $\mathbf{b}=(x,y)\in X$.
Then $\left(d\left(\mathbf{b}_n,\mathbf{b}\right)\right)$ is null. Further

$$\left|\sqrt{2}+\frac{1}{n}-x\right| = \frac{1}{2}\left(2\left|\sqrt{2}+\frac{1}{n}-x\right|\right) \leq \frac{1}{2}d\left(\mathbf{b}_n,\mathbf{b}\right)$$

from which it follows that $\left|\sqrt{2}+\frac{1}{n}-x\right| \to 0$ as $n \to \infty$, and so $x = \sqrt{2}$.
In a similar way we can show that $y = 0$, and so $(x, y) = (\sqrt{2}, 0)$, contradicting the fact that $(x, y) \in X$.
Our only conclusion is that there is no point $(x, y) \in X$ to which $(\mathbf{b}_n)$ $d$-converges, that is $(\mathbf{b}_n)$ is not $d$-convergent.
There was a strong feeling that something much less formal would be acceptable for a proof that $(\mathbf{b}_n)$ is not $d$-convergent. This was suggested:
Suppose that $(\mathbf{b}_n)$ $d$-converges to $(a, b)$. Then the only possible limit is $(\sqrt{2}, 0)$ and this is not in $X$, so the sequence cannot be $d$-convergent.
No authority could be produced at the time, but many students felt that this would likely get full or near full marks.

Finally, as $(\mathbf{b}_n)$ is $d_X$-Cauchy but not $d$-convergent, we conclude that $X$ is not $d$-complete.

(e) $X$ is not $d$-complete so it is not $d$-compact (HB, p136, Thm. 2.3).
$X$ is not $d$-compact so it is not $d$-sequentially compact (HB, p134, Thm. 5.12).

Q 7.

(a)

(i) The Sylow 7-subgroup has order $7^2 = 49$.

Given the question, "49" alone gets the mark for (i).

(ii) $|G| = 539 = 11 \times 7^2$.
We use the Sylow Theorems (Thm. 1.12 HB p. 39)
Let $m_p$ denote the number of Sylow $p$-subgroups of $G$.
We first consider the Sylow 7-subgroups: $m_7 \mid 11$, so $m_7$ is 1 or 11.
Also, $m_7 \equiv 1 \pmod 7$ and since $11 \not\equiv 1 \pmod 7$, $m_7 = 1$.
Hence there is a unique Sylow 7-subgroup.
Now consider the Sylow 11-subgroups: $m_{11} \mid 49$, so $m_{11}$ is 1, 7 or 49.
Also, $m_{11} \equiv 1 \pmod{11}$ and since neither 7 nor 49 is $\equiv 1 \pmod{11}$, $m_{11} = 1$.
Hence there is a unique Sylow 11-subgroup.

(iii) Let $H$ be the Sylow 7-subgroup and $K$ be the Sylow 11-subgroup. Then

- $|G| = 539 = 49 \times 11 = |H| \times |K|$,
- $|H|(= 49)$ and $|K|(= 11)$ are coprime,
- As shown in (ii) above, as the Sylow 7-subgroup and the Sylow 11-subgroup are unique, they are normal (by Lemma 1.9, HB p 39).

Hence by Thm. 2.3 (HB p39), $G \cong H \times K$. Further,

- $|K| = 11$ is prime, hence $K$ is cyclic and so Abelian.
- Since 7 is prime, every group of order $7^2$ is Abelian (Thm. 2.15, HB p36). Hence $H$ is Abelian.

Finally, since the direct product of two Abelian groups is also Abelian, it follows from above that $G$ is Abelian.

(b)

$$\begin{aligned} abab^3 &= a(ba)b^3 \\ &= a(ab^{-1})b^3 \quad (\text{ using (see HB, p34) } ba = ab^{-1}) \\ &= a^2b^2, \end{aligned}$$

which, having the form $a^ib^j$ with $i = 0 \ldots 3$ and $j = 0 \ldots 9$, is the required answer.

(c)

(i) As $m \mid |G|$ and $p \mid m$, it follows that $p \mid |G|$.
Also, since $p \mid |G|$, there is an element of order $p$ in $G$ (by Cauchy's Theorem). This element generates a subgroup of order $p$ and, since $G$ is Abelian, this subgroup is a normal subgroup.

(ii) Let $P(n)$ be the statement: "Given an Abelian group $G$ of order $n$, if $m$ divides $|G|$, $G$ has a subgroup of order $m$."

Clearly $P(1)$ is true: if we have an Abelian group $G$ of order 1, and $m$ divides $|G|$, then $m = 1$ and $\{e\}$ is a subgroup of $G$ with order $m(= 1)$.
Now suppose that $P(1), P(2), \ldots P(k)$ are all true. That is the result holds for all groups of order $1, 2, \ldots k$.
Take any Abelian group G of order $k + 1$ and any $m \in \mathbb{N}$ that divides $|G| = k + 1$.
If $m = 1$, the result is easy - every group has a subgroup of order 1. If $m$ is prime, it follows from c(i) (or from Cauchy's Theorem) that $G$ has a subgroup of order $m$.
This leaves the case when $m$ is composite and $m > 1$. Let $p$ be any prime factor of $m$. Then we have $m \mid |G|$ and $p \mid m$, and we may use part(i) to conclude that $G$ has a normal subgroup, $N$, of order $p$.
Then $G/N$ has order smaller than $k + 1$, which will have $\frac{m}{p}$ as a divisor (which we know is an integer since $p|m$).
Since $P(1), P(2), \ldots P(k)$ are all true, $G/N$ has a subgroup of order $\frac{m}{p}$.
Finally, by the Correspondence Theorem, $G$ will have a subgroup of order $m$.
Thus we have established $P(k + 1)$. Hence by the Second Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Q 8.

(a) If $N(r) = a^2 + b^2 = 5$, the only integer solutions are $a = \pm 1$, $b = \pm 2$ and $a = \pm 2$, $b = \pm 1$. So the only elements $a + bi$ with norm of 5 are:

$$1 \pm 2i,\ -1 \pm 2i,\ 2 \pm i,\ -2 \pm i$$

as required.

As $\mathbb{Z}[i]$ has a Euclidean norm, it is a Euclidean domain (Def 3.1, HB p.55) and the only units have Norm $N(u) = 1$ (Lemma 3.5 p. 55). Therefore the only units in $\mathbb{Z}[i]$ are $\pm 1$ and $\pm i$. If $r$ and $s$ are associates, then $r = us$ for some unit $u \in \mathbb{Z}[i]$. There are four units in $\mathbb{Z}[i]$, thus any element of $\mathbb{Z}[i]$ can have at most four associates (including itself), and so the eight elements above cannot all be associates of each other.

(b) Let $r = a + bi$, then $s = x(a + bi)$ and $t = y(a + bi)$ for some $x, y \in \mathbb{Z}[i]$. As $N$ is a multiplicative norm:

$$N(s) = N(x)\,N(a + bi) = N(x)\,N(r) \qquad \text{and}$$
$$N(t) = N(y)\,N(a + bi) = N(y)\,N(r)$$

As norms take integer values, $N(r)$ is a common factor of $N(s)$ and $N(t)$ in $\mathbb{Z}$, as required.

(c) $N(3 + i) = 9 + 1 = 10$ and $N(5) = 25$. So, from part (b), any common factor of these two elements must have norm 1 or 5. Any element with $N = 1$ is a unit, see part (a), so any common factor with $N = 5$ will be an hcf. Now:

$$\frac{5}{1 + 2i} = \frac{5(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5(1 - 2i)}{5} = 1 - 2i \in \mathbb{Z}[i]$$

and

$$\frac{3 + i}{1 + 2i} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{5 - 5i}{5} = 1 - i \in \mathbb{Z}[i]$$

Since $1 + 2i$ has norm 5, any other common factor will be a unit, or a unit multiple of 1+2i and so $1 + 2i$ is an hcf, as required.

(d) $\mathbb{Z}[i]$ is a Euclidean domain and therefore a PID (diagram, HB p.96). From Lemma 3.7, HB p.97, $I = \langle 3 + i, 5\rangle$ will also be generated by any element that is an hcf of $3 + i$ and 5. From part (c), one such element is $1 + 2i$. Any other hcf will be an associate of $1 + 2i$.

$$1(1 + 2i) = 1 + 2i$$
$$-1(1 + 2i) = -1 - 2i$$
$$i(1 + 2i) = -2 + i$$
$$-i(1 + 2i) = 2 - i$$

From part (a), there are only four associates of $1 + 2i$ in $\mathbb{Z}[i]$, so the full set of hcfs of $3 + i$ and 25 in $\mathbb{Z}[i]$ is

$$\{1 + 2i,\ -1 - 2i,\ -2 + i,\ 2 - i\}$$

and for any of these elements $d$, $I = \langle d\rangle$, as required.

(e) By Th. 4.22, HB p.99, we need to show that $1+2i$ is irreducible in $\mathbb{Z}[i]$. Since $N(1 + 2i) = 5$, any factor of $1 + 2i$ must have $N = 1$ or $N = 5$. But if $N = 1$, the proposed factor is a

unit, while if $N = 5$ it is an associate. Hence $1 + 2i$ is irreducible in $\mathbb{Z}[i]$ (definition 2.5, HB p. 54) and so $I$ is a maximal ideal, as required.

(f) Let $a + bi$ be an element of $\mathbb{Z}[i]$, then

$$\begin{aligned} \alpha + \beta i &= (a + bi)(2 - i) \\ &= 2a + b + (-a + 2b)\, i \end{aligned}$$

then

$$\begin{aligned} \alpha + 2\beta &= 2a + b - 2a + 4b \\ &= 5b \equiv 0 \pmod 5 \end{aligned}$$

hence $I \subseteq \{\alpha + \beta i : \alpha, \beta \in \mathbb{Z}, \alpha + 2\beta \equiv 0 \pmod 5\}$.

Now suppose that $\gamma + \delta i \in \{\alpha + \beta i : \alpha, \beta \in \mathbb{Z}, \alpha + 2\beta \equiv 0 \pmod 5\}$, so that $\gamma + 2\delta = 5b$ for some integer $b$. Let $a = 2b - \delta$, so that

$$\gamma = 2a + b \text{ and } \delta = -a + 2b.$$

Then

$$\begin{aligned} \gamma + \delta i &= (2a + b) + (-a + 2b)i \\ &= 2a - bi^2 - ai + 2bi \\ &= (2 - i)(a + bi) \end{aligned}$$

from which we conclude that $\gamma + \delta i \in I$ and therefore

$$\{\alpha + \beta i : \alpha, \beta \in \mathbb{Z}, \alpha + 2\beta \equiv 0 \pmod 5\} \subseteq I.$$

Thus $I = \{\alpha + \beta i : \alpha, \beta \in \mathbb{Z}, \alpha + 2\beta \equiv 0 \pmod 5\}$ as required.

Q 9.

(a)

(i) $d(1,5) = 2$ (as $1-5$ is a non-zero even number).

(ii)

i. From the given definition, $d(a,b) \geq 0$ for each $a, b \in \mathbb{N}$.
If $a = b$, that is $a - b = 0$, then $d(a,b) = 0$ immediately. If $d(a,b) = 0$, then the only possibility is when $a - b = 0$, or when $a = b$ from the definition.
Hence (M1) holds.

ii. We note that $a - b$ is 0 if and only if $b - a$ is 0 (and so $a - b$ is non-zero if and only if $b - a$ is non-zero). Also $a - b$ is even if and only if $b - a$ is even (and so $a - b$ is odd if and only if $b - a$ is odd). It therefore follows that $d(a,b) = d(b,a)$, for all $a, b \in \mathbb{N}$. That is, (M2) holds.

iii. To show (M3) holds, take any $a, b, c \in \mathbb{N}$. We want to show that

$$d(a,c) \leq d(a,b) + d(b,c) \text{ for all } a, b, c \in \mathbb{N}$$

Consider cases.

1. if $a = c$, the LHS $= 0$ and so the result follows easily, as RHS $\geq 0$.
2. if $a = b$, the LHS = RHS and again the result follows easily. Similarly, if $b = c$.
3. Thus we are reduced to the case where $a, b, c$ are all distinct. But in this case, the LHS is at most 3 while the RHS is at least 4, and the result follows. Thus (M3) holds.

So by Definition 2.1, HB p72, $d$ is a metric on $\mathbb{N}$.

(b) Here:
$A = \left\{ f \in C[0,1] : |f(x) - f(y)| \leq \sqrt{|x-y|} \text{ for each } x, y \in [0,1] \right\}$

and $B = \{f \in C[0,1] : |f(x)| \leq 1 \text{ for } x \in [0,1]\}$.

(i) Let $(f_n)$ be a sequence in $A$ such that $f_n$ $d_{\max}$-converges to $f$. We must show that $f \in A$. That is, we must show that,

$$|f(x) - f(y)| \leq \sqrt{|x-y|}, \text{ for each } x, y \in [0, 1]$$

Now, by the Triangle Inequality, for each $x, y \in [0, 1]$

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq d_{\max}(f_n, f) + \sqrt{|x-y|} + d_{\max}(f_n, f) \end{aligned}$$

since $f_n \in A$. This inequality is true for arbitrary $n$ and $d_{\max}(f_n, f) \to 0$ as $n \to \infty$, so

$$|f(x) - f(y)| \leq 0 + \sqrt{|x-y|} + 0, \text{ for each } x, y \in [0, 1]$$

and so $f \in A$. Hence $A$ is $d_{\max}$-closed.

(ii) Let $f_0$ denote the zero function.
By the definition of the set $B$

$$f \in B \iff |f(x) - 0| \leq 1, \quad \text{for all } x \in [0,1] \iff d_{\max}(f,\, f_0) \leq 1$$

so $B = B_{d_{\max}}[f_0,\, 1]$ which is a closed ball and therefore a closed set in $(C[0,1],\, d_{\max})$ by Appendix 5, HB p89.

Thus $A$ and $B$ are both $d_{\max}$-closed and so their intersection $A \cap B$ is $d_{\max}$-closed by Proposition 1.17, HB p81.

(iii) Since $A \cap B \subseteq C[0,\, 1]$, we use the Arzelà-Ascoli Theorem to show that $A \cap B$ is $d_{\max}$-sequentially compact.

1. From part (ii) above, $A \cap B$ is $d_{\max}$-closed

2. From example 1. of pointwise bounded sets with $g(x) = 1$, HB p130, $B$ is a pointwise bounded set and so $A \cap B \subseteq B$ is also a pointwise bounded set of functions.

3. Take $\varepsilon > 0$. If $x, y \in [0,1]$ and $|x - y| < \delta$, then for each $f \in A$,

$$|f(x) - f(y)| \leq \sqrt{|x - y|} < \sqrt{\delta}$$

In particular, if $\delta = \varepsilon^2$, then
$\sqrt{|x - y|} \leq \sqrt{\delta} \leq \sqrt{(\varepsilon^2)} \leq \varepsilon$
then for each $f \in A$ and each $x, y \in [0,1]$,
$|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [0,1]$ satisfies $|x - y| < \delta$. So $A$ is an equicontinuous set of functions.
Therefore, $A \cap B \subseteq A$ is also an equicontinuous set of functions by example 2., HB p131.

Thus all three conditions of the Arzelà-Ascoli Theorem are satisfied and so $A \cap B$ is $d_{\max}$-sequentially compact.